Reconstruction of invariants of configuration spaces of hyperbolic curves from associated Lie algebras

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Combinatorial Anabelian Geometry and Related Topics

2021/07/06

Table of contents

§1 Introduction

K: algebraically closed field of characteristic zero

X: hyperbolic curve$/K$ of type (g, r)

$X_n := \{(x_1, \ldots, x_n) \in X \times_K \cdots \times_K X \mid x_i \neq x_j \ (\forall i \neq j)\}$
: n-th configuration space of X

$\Pi_n^\Sigma := \pi_1(X_n)^\Sigma$: maximal pro-$\Sigma$ quotient of $\pi_1(X_n)$
($\Sigma = \{l\}$ for $^\exists l$ or Σ: the set of all prime numbers)

We shall refer to a profinite group isom to Π_1^Σ (for some X) as a surface group.

Theorem A (Hoshi-Minamide-Mochizuki (preprint,2017))

$\Pi_n^{\Sigma} \overset{\text{recon.}}{\rightsquigarrow} n, \underline{g, r}$, "generalized fiber subgroups"

if $n \geq 2$

Theorem B (S. (preprint,2018))

$\mathrm{Gr}(\Pi_n^l) \overset{\text{recon.}}{\rightsquigarrow} n, \underline{g, r}$, "generalized fiber ideals"

if $n \geq 2$

(More precisely, there are algorithms of reconstructing these invariants from the abstract Lie algebra over $\mathbb{Z}_l$ obtained by forgetting the grading of $\mathrm{Gr}(\Pi_n^l)$.)

My research on Theorem B was motivated by [HMM].

Today, we will compare their algorithms to observe similarities (i.e., what can be applied commonly to both) and differences.

Definition (generalized projection)

$p : X_n \to X_m \ (0 \leq m \leq n)$

is a generalized projection morphism $\overset{\text{def}}{\Leftrightarrow}$

- If $(g, r) \notin \{(0,3), (1,1)\}$, then
  $p : X_n \to X_m$: projection morphism
- If $(g, r) = (0, 3)$, then $X_n \cong (\mathcal{M}_{0,n+3})_K$
  $p : X_n \xrightarrow{\sim} (\mathcal{M}_{0,n+3})_K \to (\mathcal{M}_{0,m+3})_K \xrightarrow{\sim} X_m$
- If $(g, r) = (1, 1)$, then $X_n \cong E_{n+1}/E \ \ (E := X^{\mathrm{cpt}})$
  $p : X_n \xrightarrow{\sim} E_{n+1}/E \to E_{m+1}/E \xrightarrow{\sim} X_m$

Definition (generalized fiber subgroup)

A generalized projection morphism induces $\Pi_n^{\Sigma} \twoheadrightarrow \Pi_m^{\Sigma}$.

$\ker(\Pi_n^{\Sigma} \twoheadrightarrow \Pi_m^{\Sigma})$: generalized fiber subgroup

(of co-length m)

$\mathrm{GFS}_m(\Pi_n^{\Sigma})$: the set of gen. fiber subgps of co-length m

Note: $N \in \mathrm{GFS}_m(\Pi_n^{\Sigma})$ is isomorphic to "Π_{n-m}^{Σ}" of a hyperbolic curve of type $(g, r+m)$.

Definition (Lie algebra associated to X_n)

Write $\Pi_n^l(1) := \Pi_n^l (:= \Pi_n^{\{l\}})$,

$\Pi_n^l(2) := \ker(\Pi_n^l \twoheadrightarrow (\pi_1(X^{\mathrm{cpt}} \times_K \cdots \times_K X^{\mathrm{cpt}})^l)^{\mathrm{ab}})$,

$\Pi_n^l(m) := \overline{\langle [\Pi_n^l(m_1), \Pi_n^l(m_2)]} \mid m_1 + m_2 = m \rangle \ (m \geq 3)$,

$\mathrm{Gr}^m(\Pi_n^l) := \Pi_n^l(m)/\Pi_n^l(m+1)$,

$\mathrm{Gr}(\Pi_n^l) := \bigoplus_{m \geq 1} \mathrm{Gr}^m(\Pi_n^l)$.

Then $\mathrm{Gr}(\Pi_n^l)$: graded Lie algebra over $\mathbb{Z}_l$.

We shall refer to an abstract Lie algebra$/\mathbb{Z}_l$ isomorphic to $\mathrm{Gr}(\Pi_1^l)$ as a surface algebra.

Definition (generalized fiber ideal)

A (generalized) projection morphism $X_n \to X_m$

induces $\mathrm{Gr}(\Pi_n^l) \twoheadrightarrow \mathrm{Gr}(\Pi_m^l)$.

$\ker(\mathrm{Gr}(\Pi_n^l) \twoheadrightarrow \mathrm{Gr}(\Pi_m^l))$: (generalized) fiber ideal (of co-length m)

$\mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^l))$: the set of gen. fib. ideals of co-length m

Note: $\mathfrak{i} \in \mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^l))$ is isomorphic to "$\mathrm{Gr}(\Pi_{n-m}^l)$" of a hyperbolic curve of type $(g, r+m)$.

$\mathrm{Gr}(\Pi_n^l)$ has a presentation with generators

$X_i^{(k)}, Y_i^{(k)} \in \mathrm{Gr}^1(\Pi_n^l)$, $Z_j^{(k)}, W_h^{(k)} \in \mathrm{Gr}^2(\Pi_n^l)$

$(1 \leq i \leq g,\ 1 \leq j \leq r,\ 1 \leq k, h \leq n)$

and relations (R1–10):

$$\sum_{i=1}^{g}[X_i^{(k)}, Y_i^{(k)}] + \sum_{j=1}^{r} Z_j^{(k)} + \sum_{h=1}^{n} W_h^{(k)} = 0, \qquad \text{(R1)}$$

$$W_k^{(k)} = 0, \qquad \text{(R2)}$$

$$W_h^{(k)} = W_k^{(h)}, \qquad \text{(R3)}$$

$$[X_i^{(k)}, X_{i'}^{(k')}] = [Y_i^{(k)}, Y_{i'}^{(k')}] = 0 \quad (k \neq k'), \tag{R4}$$

$$[X_i^{(k)}, Y_{i'}^{(k')}] = 0 \quad (i \neq i', k \neq k'), \tag{R5}$$

$$[X_i^{(k)}, Y_i^{(k')}] = W_k^{(k')} \quad (k \neq k'), \tag{R6}$$

$$[X_i^{(k)}, Z_j^{(k')}] = [Y_i^{(k)}, Z_j^{(k')}] = 0 \quad (k \neq k'), \tag{R7}$$

$$[Z_j^{(k)}, Z_{j'}^{(k')}] = 0 \quad (j \neq j', k \neq k'), \tag{R8}$$

$$[X_i^{(k)}, W_h^{(k')}] = [Y_i^{(k)}, W_h^{(k')}] = [Z_j^{(k)}, W_h^{(k')}] = 0 \quad (k \notin \{k', h\}), \tag{R9}$$

$$[W_h^{(k)}, W_{h'}^{(k')}] = 0 \quad (\{k, h\} \cap \{k', h'\} = \emptyset). \tag{R10}$$

$$
\left.\begin{array}{cc|cc|cc|cccc}
X_1^{(1)} \cdots & X_g^{(1)} & Y_1^{(1)} \cdots & Y_g^{(1)} & Z_1^{(1)} \cdots & Z_r^{(1)} & 0 & W_2^{(1)} & \cdots & W_n^{(1)} \\
X_1^{(2)} \cdots & X_g^{(2)} & Y_1^{(2)} \cdots & Y_g^{(2)} & Z_1^{(2)} \cdots & Z_r^{(2)} & W_1^{(2)} & 0 & \cdots & W_n^{(2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
X_1^{(n)} \cdots & X_g^{(n)} & Y_1^{(n)} \cdots & Y_g^{(n)} & Z_1^{(n)} \cdots & Z_r^{(n)} & W_1^{(n)} & W_2^{(n)} & \cdots & 0
\end{array}\right.
$$

§2 Reconstruction algorithms

- $\Pi_n^{\Sigma} \overset{\text{recon.}}{\rightsquigarrow} n$

Proposition 1

Let $H \subset \Pi_n^l$: closed subgroup isom to $\mathbb{Z}_l^{\oplus s}$.

Then $s \leq n$.

(Proof) $\underline{n = 1}$: a property of pro-l surface groups

$\underline{n \geq 2}$: consider $1 \to \Pi_{n/n-1}^l \to \Pi_n^l \to \Pi_{n-1}^l \to 1$

($\Pi_{n/n-1}^l$: surface group) □

Theorem 2

$^{\exists}H \subset \Pi_n^l$: closed subgroup isom to $\mathbb{Z}_l^{\oplus n}$.

(This follows from the existence of log-full points of the log configuration space of a suitable stable log curve.)

Algorithm:

Take l: a prime number s.t. $(\Pi_n^\Sigma)^l \neq \{1\}$ $(\Leftrightarrow l \in \Sigma)$

$\Pi_n^l = (\Pi_n^\Sigma)^l$

$n = \max\{s \in \mathbb{Z}_{\geq 0} \mid {}^{\exists}H$: closed subgp of Π_n^l s.t. $H \cong \mathbb{Z}_l^{\oplus s}\}$

· $\mathrm{Gr}(\Pi_n^l) \overset{\text{recon.}}{\rightsquigarrow} n$

Proposition 3

Let $\mathfrak{a} \subset \mathrm{Gr}(\Pi_n^l)$: abelian subalgebra/$\mathbb{Z}_l$ isom to $\mathbb{Z}_l^{\oplus s}$ as a $\mathbb{Z}_l$-module. Then $s \leq n$.

(Lie alg. $\mathfrak{L}$ is abelian $\overset{\text{def}}{\Leftrightarrow}$ ${}^\forall a, b \in \mathfrak{L}\ [a, b] = 0$)

(Proof) $\underline{n = 1}$: $[a, b] = 0 \Rightarrow\ a, b$: linearly dependent

$\underline{n \geq 2}$: consider

$$1 \to \mathrm{Gr}(\Pi_{n/n-1}^l) \to \mathrm{Gr}(\Pi_n^l) \to \mathrm{Gr}(\Pi_{n-1}^l) \to 1 \qquad \square$$

Theorem 4

${}^{\exists}\mathfrak{a} \subset \mathrm{Gr}(\Pi_n^l)$: abelian subalg. isom to $\mathbb{Z}_l^{\oplus n}$.

(Proof)

If $g > 0$, then $[X_1^{(h)}, X_1^{(k)}] = 0$ for $1 \leq h < k \leq n$.

If $r > 0$, then write $A^{(k)} := Z_r^{(k)} + \sum_{h=1}^{k-1} W_h^{(k)}$.

Then $[A^{(h)}, A^{(k)}] = 0$ for $1 \leq h < k \leq n$. □

Algorithm:

$$n = \max\{s \in \mathbb{Z}_{\geq 0} \mid {}^{\exists}\mathfrak{a} : \text{abelian subalg. of } \mathrm{Gr}(\Pi_n^l) \text{ s.t. } \mathfrak{a} \cong \mathbb{Z}_l^{\oplus s}\}$$

- $\Pi_n^\Sigma \overset{\text{recon.}}{\rightsquigarrow} \mathrm{GFS}_1(\Pi_n^\Sigma)$

Theorem 5

Let $H \subset \Pi_n^\Sigma$: normal closed subgroup such that Π_n^Σ / H is isom to a surface group which is not free of rank 2.
Then $^\exists N \in \mathrm{GFS}_1(\Pi_n^\Sigma)$ s.t. $N \subset H$.

In particular, $(g, r) \notin \{(0,3),(1,1)\} \Leftrightarrow {}^\exists H$ as above.

Proposition 6

$\Pi_n^{l,\mathrm{ab}}$: free $\mathbb{Z}_l$-module,

$$\mathrm{rank}_{\mathbb{Z}_l} \Pi_n^{l,\mathrm{ab}} = \begin{cases} n(r-1) + n(n-1)/2 & (g=0) \\ 2gn & (r=0) \\ n(2g+r-1) & (g,r>0). \end{cases}$$

In particular, $(g,r) = (1,1) \Leftrightarrow \mathrm{rank}_{\mathbb{Z}_l} \Pi_n^{l,\mathrm{ab}} = 2n$.

Algorithm:

$S := \{H \overset{\text{cl.}}{\triangleleft} \Pi_n^\Sigma \,|\, \Pi_n^\Sigma / H$: surf. gp, not free of rank $2\}$

If $S \neq \emptyset$ ($\Leftrightarrow (g, r) \notin \{(0,3), (1,1)\}$), then

$\mathrm{GFS}_1(\Pi_n^\Sigma)$: minimal elements of S.

$S = \emptyset \cdots$ we omit the details

($(g, r) = (1, 1) \Leftrightarrow \mathrm{rank}_{\mathbb{Z}_l} \Pi_n^{l,\mathrm{ab}} = 2n$,

construct Lie alg. isom to $\mathrm{Gr}(\Pi_n^l)$ ($\cong \mathrm{Gr}^{\mathrm{lcs}}(\Pi_n^l)$),

consider $\Pi_n^\Sigma \to \Pi_n^{l,\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \ldots$)

· $\mathrm{Gr}(\Pi_n^l) \overset{\text{recon.}}{\rightsquigarrow} \mathrm{GFI}_1(\mathrm{Gr}(\Pi_n^l))$

Theorem 7

We can classify all (not necessarily graded) surjective homomorphisms/$\mathbb{Z}_l$ from $\mathrm{Gr}(\Pi_n^l)$ to a surface algebra/$\mathbb{Z}_l$.

(Proof: direct calculation)

Corollary 8

Let $\mathfrak{i} \subset \mathrm{Gr}(\Pi_n^l)$: an ideal$/\mathbb{Z}_l$ (not necessarily graded).
Then $\mathfrak{i} \in \mathrm{GFI}_1(\mathrm{Gr}(\Pi_n^l)) \Leftrightarrow \mathrm{Gr}(\Pi_n^l)/\mathfrak{i} \cong \mathrm{Gr}(\Pi_1^l)$.

Corollary 9

$\mathrm{Gr}(\Pi_1^l)$: unique surface algebra $\mathfrak{s}$ (up to isom) s.t.

- $\mathrm{Gr}(\Pi_n^l) \overset{\exists}{\twoheadrightarrow} \mathfrak{s}$,
- for any surface alg. $\mathfrak{h}$, if $\mathrm{Gr}(\Pi_n^l) \overset{\exists}{\twoheadrightarrow} \mathfrak{h}$, then $\mathfrak{s} \overset{\exists}{\twoheadrightarrow} \mathfrak{h}$.

Algorithm:

(isom class of) $\mathrm{Gr}(\Pi_1^l)$:

(unique) surface algebra $\mathfrak{s}$ satisfying

- $\mathrm{Gr}(\Pi_n^l) \overset{\exists}{\twoheadrightarrow} \mathfrak{s}$,
- for any surface alg. $\mathfrak{h}$, if $\mathrm{Gr}(\Pi_n^l) \overset{\exists}{\twoheadrightarrow} \mathfrak{h}$, then $\mathfrak{s} \overset{\exists}{\twoheadrightarrow} \mathfrak{h}$.

$$\mathrm{GFI}_1(\mathrm{Gr}(\Pi_n^l)) = \{\mathfrak{i} \overset{\text{ideal}/\mathbb{Z}_l}{\subset} \mathrm{Gr}(\Pi_n^l) \,|\, \mathrm{Gr}(\Pi_n^l)/\mathfrak{i} \cong \mathrm{Gr}(\Pi_1^l)\}$$

$\cdot\ \Pi_n^{\Sigma} \overset{\text{recon.}}{\rightsquigarrow} \mathrm{GFS}_m(\Pi_n^{\Sigma}),\ \mathrm{Gr}(\Pi_n^{l}) \overset{\text{recon.}}{\rightsquigarrow} \mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^{l}))$

Algorithm:

$$\mathrm{GFS}_m(\Pi_n^{\Sigma}) = \bigcup_{H \in \mathrm{GFS}_{m-1}(\Pi_n^{\Sigma})} \mathrm{GFS}_1(H)$$

$$\mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^{l})) = \bigcup_{\mathfrak{i} \in \mathrm{GFI}_{m-1}(\mathrm{Gr}(\Pi_n^{l}))} \mathrm{GFI}_1(\mathfrak{i})$$

(We set $\mathrm{GFS}_1(\{1\}) := \emptyset,\ \mathrm{GFI}_1(\{0\}) := \emptyset$.)

Remark

・We do not use "n" for $\mathrm{Gr}(\Pi_n^l) \overset{\text{recon.}}{\rightsquigarrow} \mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^l))$. So we can obtain other algorithms for $\mathrm{Gr}(\Pi_n^l) \overset{\text{recon.}}{\rightsquigarrow} n$.

For example, n: unique nonnegative integer m s.t. $\mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^l)) = \{\{0\}\}$.

・We can reconstruct $\mathrm{GFS}_1(\Pi_n^\Sigma)$ from Π_n^Σ in a similar manner to $\mathrm{Gr}(\Pi_n^l) \overset{\text{recon.}}{\rightsquigarrow} \mathrm{GFI}_1(\mathrm{Gr}(\Pi_n^l))$.

- $\Pi_n^\Sigma \overset{\text{recon.}}{\rightsquigarrow} (g, r)$ (if $n \geq 2$)

Proposition 10

- $\mathrm{rank}_{\mathbb{Z}_l} \Pi_1^{l,\mathrm{ab}} = 2g + \max\{r-1, 0\}$
- Π_1^l: free $\Leftrightarrow r > 0$

Theorem 11 ([CbTpl] Lemma 1.3(iv))

Suppose: $r > 0$.

Consider the natural action $\Pi_1^l \to \mathrm{Aut}(\Pi_{2/1}^{l,\mathrm{ab}})$ determined by $1 \to \Pi_{2/1}^l \to \Pi_2^l \to \Pi_1^l \to 1$.

Then $\ker(\Pi_1^l \to \mathrm{Aut}(\Pi_{2/1}^{l,\mathrm{ab}})) = \ker(\Pi_1^l \twoheadrightarrow \pi_1(X^{\mathrm{cpt}})^{l,\mathrm{ab}})$.

Algorithm(1/2): (Suppose: $n \geq 2$.)

Take $H \in \mathrm{GFS}_{n-2}(\Pi_n^l),\ N \in \mathrm{GFS}_1(H)$.
$(\Pi_2^l \cong \Pi_n^l/H,\ \Pi_1^l \cong H/N)$

If H/N: not free $(\Leftrightarrow r = 0)$, then
$g = \frac{1}{2}\mathrm{rank}_{\mathbb{Z}_l}(H/N)^{\mathrm{ab}},\ r = 0$.

Algorithm(2/2):

If H/N: free, then, since

$$\ker(\Pi_1^l \twoheadrightarrow \pi_1(X^{\mathrm{cpt}})^{l,\mathrm{ab}}) = \ker(H/N \to \mathrm{Aut}(N^{\mathrm{ab}})),$$

$$\ker(\Pi_1^{l,\mathrm{ab}} \twoheadrightarrow \pi_1(X^{\mathrm{cpt}})^{l,\mathrm{ab}})$$

$$= \mathrm{Im}(\ker(H/N \to \mathrm{Aut}(N^{\mathrm{ab}})) \to (H/N)^{\mathrm{ab}}).$$

$$r = \mathrm{rank}_{\mathbb{Z}_l}(\ker(\Pi_1^{l,\mathrm{ab}} \twoheadrightarrow \pi_1(X^{\mathrm{cpt}})^{l,\mathrm{ab}})) + 1$$

$$g = \frac{1}{2}(\mathrm{rank}_{\mathbb{Z}_l}(H/N)^{\mathrm{ab}} - r + 1)$$

· $\mathrm{Gr}(\Pi_n^l) \overset{\text{recon.}}{\rightsquigarrow} (g, r)$

Write $\mathrm{Gr}(\Pi_n^l)(2) := \bigoplus_{m \geq 2} \mathrm{Gr}^m(\Pi_n^l)$,

$$\mathrm{Gr}(\Pi_n^l)[1] := \mathrm{Gr}(\Pi_n^l),$$
$$\mathrm{Gr}(\Pi_n^l)[m] := [\mathrm{Gr}(\Pi_n^l), \mathrm{Gr}(\Pi_n^l)[m-1]].$$

Theorem 12

Suppose: $n \geq 2$ and $g > 0$.

Let $\mathfrak{i}, \mathfrak{j}$: distinct elements of $\mathrm{GFI}_{n-1}(\mathrm{Gr}(\Pi_n^l))$.

Then $\mathfrak{i} \cap \mathrm{Gr}(\Pi_n^l)(2) = \{a \in \mathfrak{i} \mid {}^{\forall} b \in \mathfrak{j} \;\; [a, b] \in \mathrm{Gr}(\Pi_n^l)[3]\}$.

(Proof: if $j \neq j'$ and $k \neq k'$,

$[Z_j^{(k)}, X_i^{(k')}] = [Z_j^{(k)}, Y_i^{(k')}] = [Z_j^{(k)}, Z_{j'}^{(k')}] = 0,$

$[Z_j^{(k)}, W_h^{(k')}] = [Z_j^{(k)}, [X_1^{(h)}, Y_1^{(k')}]] \in \mathrm{Gr}(\Pi_n^l)[3],$

$[Z_j^{(k)}, Z_j^{(k')}] = -[Z_j^{(k)}, W_k^{(k')}] \in \mathrm{Gr}(\Pi_n^l)[3].$)

Proposition 13

- $\mathrm{rank}_{\mathbb{Z}_l}(\mathrm{Gr}(\Pi_1^l)/\mathrm{Gr}(\Pi_1^l)[2]) = 2g + \max\{r-1, 0\}$
- $\mathrm{Gr}(\Pi_1^l)$: free $\Leftrightarrow r > 0$

It suffices to determine whether $g = 0$ or not.

Idea: If $g = 0$, there are many homomorphisms determined as follows:

$X \hookrightarrow Y$: of type $(0, 3)$,

$p : Y_n \to Y$: a gen. proj. which is not a proj.,

$\mathrm{Gr}(\Pi_n^l) \twoheadrightarrow \mathrm{Gr}(\pi_1(Y)^l)$: determined by $\underline{X_n \hookrightarrow Y_n \xrightarrow{p} Y}$.

(exceptional mor.)

Theorem 14

Write $E := \{\mathfrak{i} \overset{\text{ideal}/\mathbb{Z}_l}{\subset} \mathrm{Gr}(\Pi_n^l) \,|\, \mathrm{Gr}(\Pi_n^l)/\mathfrak{i}$: free of rank 2, ${}^\forall \mathfrak{j} \in \mathrm{GFI}_1(\mathrm{Gr}(\Pi_n^l))\ \mathfrak{j} \not\subset \mathfrak{i}\}$.

$$\text{Then } \sharp E = \begin{cases} \binom{r}{2} \cdot \binom{n}{2} + r \cdot \binom{n}{3} + \binom{n}{4} & (g = 0, r \neq 3) \\ \binom{n}{2} & (g = 1, r \neq 1) \\ 0 & (\text{else}), \end{cases}$$

$$\sharp\, \mathrm{GFI}_1(\mathrm{Gr}(\Pi_n^l)) = \begin{cases} \binom{n+3}{4} & ((g,r) = (0,3)) \\ \binom{n+1}{2} & ((g,r) = (1,1)) \\ n & (\text{else}). \end{cases}$$

Algorithm(1/2): (Suppose: $n \geq 2$.)

Write $E := \{\mathfrak{i} \overset{\text{ideal}/\mathbb{Z}_l}{\subset} \mathrm{Gr}(\Pi_n^l) \,|\, \mathrm{Gr}(\Pi_n^l)/\mathfrak{i}$: free of rank 2, $\forall \mathfrak{j} \in \mathrm{GFI}_1(\mathrm{Gr}(\Pi_n^l))\ \mathfrak{j} \not\subset \mathfrak{i}\}$.

(Since free Lie algebra$/\mathbb{Z}_l$ of rank 2 is a surface algebra, we can reconstruct E by Theorem 7.)

If $\sharp E \leq \binom{n}{2}$ and $\sharp\, \mathrm{GFI}_1(\mathrm{Gr}(\Pi_n^l)) < \binom{n+3}{4}$ $(\Leftrightarrow g > 0)$, then take $\mathfrak{i}, \mathfrak{j}$: distinct elements of $\mathrm{GFI}_{n-1}(\mathrm{Gr}(\Pi_n^l))$.

$$\mathfrak{i} \cap \mathrm{Gr}(\Pi_n^l)(2) = \{a \in \mathfrak{i} \,|\, \forall b \in \mathfrak{j}\ \ [a, b] \in \mathrm{Gr}(\Pi_n^l)[3]\}$$
$$g = \frac{1}{2}\mathrm{rank}_{\mathbb{Z}_l}(\mathfrak{i}/\mathfrak{i} \cap \mathrm{Gr}(\Pi_n^l)(2))$$

Algorithm(2/2):

If $\sharp E > \binom{n}{2}$ or $\sharp\,\mathrm{GFI}_1(\mathrm{Gr}(\Pi_n^l)) = \binom{n+3}{4}$,

then $g = 0$.

If $\mathrm{Gr}(\Pi_1^l)$: free ($\Leftrightarrow r > 0$), then

$r = \mathrm{rank}_{\mathbb{Z}_l}(\mathrm{Gr}(\Pi_1^l)/\,\mathrm{Gr}(\Pi_1^l)[2]) + 1 - 2g.$

If $\mathrm{Gr}(\Pi_1^l)$: not free, then

$r = 0.$